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On stationarity and the existence of moments in the periodic asymmetric power $GARCH$ model

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Abstract

In this article, we examine the strict and second order periodic stationarities, the existence of higher order moments and the covariance structure of the periodic asymmetric power $GARCH(p, q)$ process under general and tractable assumptions.

keywords: Periodic asymmetric power $GARCH$ model, stationarity, ergodicity, higher order moments.

1 INTRODUCTION

Autoregressive conditionally heteroskedastic ($ARCH$) models were introduced by Engle[8] and their generalized $ARCH$ ($GARCH$) extension is due to Bollerslev[3]. In this models, the key concept is the conditional variance, that is, the variance conditional on the past. In the classical $GARCH$ models, the conditional variance is expressed as a linear function of the squared past value of the series. The symmetry property of standard $GARCH$ models has the following interpretation in terms of autocorrelations. $PGARCH$ model have proposed by Bollerslev and Ghysels[4] was designed to take into account the periodic dependencies in the conditional variance by allowing the parameters of the model to vary over the cycle. The probabilistic structure and the asymptotic properties of $PGARCH$ models have been developed recently as in Bibi and Aknouche[1]; [2] and Lee and Shin[12]. The Asymmetric Power $GARCH$ ($APGARCH$) model allows a wider class of power transformations than simply taking the absolute value or squaring the data as in classical heteroskedastic models. $APGARCH$ model introduced by Ding et al[7], this model depends on endogenous estimation of the

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optimal power transformation and the structure availability significant improvement over the classical $GARCH$ structure. Ding et al[7] suggest an extension of the $GARCH$ family models analyses a wider class of power transformations than simply taking the absolute value or squaring the data as in the traditional heteroskedastic models and examined the volatility of S&P 500 index returns. Consider a periodic $APGARCH\,(p,q)$ process $(\varepsilon_t)_{t\in\mathbb{Z}}$ with period $s>0$, defined on some probability space $(\Omega,\Im,P)$ by the nonlinear periodic difference equation

$$\begin{cases} \varepsilon_{st+v} = h_{st+v}e_{st+v} \\ h^{\delta}_{st+v} = \alpha_0(v) + \sum\limits_{i=1}^{q}\alpha_i(v)\left(|\varepsilon_{st+v-i}| - \beta_i(v)\varepsilon_{st+v-i}\right)^{\delta} + \sum\limits_{j=1}^{p}\gamma_j\left(v\right)h^{\delta}_{st+v-j},\ t\in\mathbb{Z} \end{cases} \tag{1.1}$$

where $(e_t)_{t\in\mathbb{Z}}$ is a sequence of independent and identically distributed $(i.i.d.)$ random variables defined on a probability space $(\Omega,\Im,P)$ with mean zero and variance unity and assuming $\mu_\delta = E\left\{e_0^\delta\right\} < \infty$. In (1.1) ε_{st+v} refers to ε_t during the $v-th$ " season", $v\in\{1,...,s\}$, of the period t. The parameters $\alpha_0(v)$, $\alpha_i(v)$, $\beta_i(v)$ and $\gamma_j(v)$ with $i\in\{1,...,q\}$ and $j\in\{1,...,p\}$ are the model coefficients at season v such that for all $v\in\{1,...,s\}$, $\alpha_0(v)>0$, $\alpha_i(v)\geq 0$, $|\beta_i\left(v\right)|\leq 1$, $\gamma_j\left(v\right)\geq 0$ and $\delta>0$ with $i\in\{1,...,q\}$, $j\in\{1,...,p\}$, the variable h_t is always strictly positive. Furthermore, we assume that ε_l is independent of e_t for $l<t$. The process $(\varepsilon_t)_{t\in\mathbb{Z}}$ is globally non stationary, but is stationary within each period, are becoming an appealing tool for investigating both volatility and distinct "seasonal" patterns and have been applied in various disciplines such as finance and monetary economics.

Remark 1.1. **1.** *If $\delta=2$ and $\beta_i(v)=0$ for all i, we get the $PGARCH\,(p,q)$ model (e.g.,Bollerslev and Ghysels[4] and Bibi and Aknouche[1]; [2]).*

2. *If $\delta=1$, we get the $PTGARCH\,(p,q)$ (e.g., Glosten et al[11] for single regime).*

3. *If $\delta=1$, $|\beta_i\left(v\right)|=1$ for all i, we get the $PGARCH\,(p,q)$ or $PARCH\,(p)$ (e.g., Bollerslev and Ghysels[4] and Bibi and Aknouche[1]; [2] and Lescheb[13]).*

4. *If $\delta\longrightarrow 0$, $\beta_i(v)=0$ for all i and using $\log h_{st+v} = \lim\limits_{\delta\longrightarrow 0}\frac{h^{\delta}_{st+v}-1}{\delta}$ one can interpret the $P\log GARCH\,(p,q)$ model is obtained as the limiting case of the $P-APGARCH\,(p,q)$.*

In this article, we focus on studying the fundamental probabilistic properties of the $P-APGARCH$ process. In Section 2, we derive some sufficient conditions for strict stationarity. In Section 3, the existence of higher-order moments are given and covariance structure. We conclude in Section 4.

Some notations are used throughout the article: $I_{(n)}$ is the $n\times n$ identity matrix. $O_{(n,,m)}$ denotes the matrix of order $n\times m$ whose entries are zeros, for simplicity we set $O_{(n)} := O_{(n,n)}$ and $\underline{O}_{(n)} := O_{(n,1)}$. The spectral radius of squared matrix M is noted $\rho\left(M\right)$, $\|.\|$ refers to the standard norm in $\mathbb{R}^n$ or the uniform induced norm in the space $\mathcal{M}(n)$ of $n\times n$ matrices, $\otimes$ denotes the

Kronecker product of matrices. $Vec(M)$ is the usual column stacking vector of the matrix M. For any $p \geq 1$, $\mathbb{L}_p = \mathbb{L}_p(\Omega, \Im, P)$ denotes the Hilbert space of random variables X defined on the probability space $(\Omega, \Im, P)$ such that $\|X_p\| = (E\{|X|^p\})^{\frac{1}{p}} < +\infty$.

2 STRICT STATIONARITY

Let $n = \max(p, q)$, the $P - APGARCH(p, q)$ process given by (1.1) can be rewritten as

$$\underline{X}_{st+v} = B(v)\,\underline{X}_{st+v-1} + \underline{\xi}(v)\,\varepsilon^{\delta}_{st+v-1} + \alpha_0(v)\underline{H} \tag{2.1}$$

where

$$B(v) := \begin{pmatrix} \underline{\gamma}_{1:n-1}(v) & I_{(n-1)} \\ \gamma_n(v) & \underline{O}'_{(n-1)} \end{pmatrix}_{n\times n},\ \underline{\xi}(v) := \begin{pmatrix} \underline{\xi}_{1:n-1}(v) \\ \xi_n(v) \end{pmatrix}_{n\times 1},$$

$$\underline{\psi}_{1:n}(v) := \begin{pmatrix} \psi_1(v) \\ \vdots \\ \psi_n(v) \end{pmatrix}_{n\times 1},\ \underline{H} := \begin{pmatrix} 1 \\ \underline{O}_{(n-1)} \end{pmatrix}_{n\times 1}$$

and $\xi_j(v) = (\pm 1 - \beta_j(v))^{\delta}\,\alpha_j(v)$ for all $1 \leq j \leq n$, and in $B(v)$ or $\underline{\xi}(v)$, $\alpha_i(v) = 0$ for $q+1 \leq i \leq n$ or $\gamma_j(v) = 0$ for $p+1 \leq j \leq n$. The k^{th} component of state vector $\underline{X}_{st+v}$ denoted by $X_{k,t}(v)$, is given by

$$X_{k,t}(v) = \begin{cases} h^{\delta}_{st+v} \text{ if } k = 1 \\ \sum\limits_{i=k}^{q} \alpha_i(v-i+k)\left(|\varepsilon_{st+v-i+k-1}| - \beta_i(v-i+k)\varepsilon_{st+v-i+k-1}\right)^{\delta} \\ + \sum\limits_{j=k}^{p} \gamma_j(v-j+k)\,h^{\delta}_{st+v-j+k-1} \text{ if } 2 \leq k \leq n \end{cases}$$

Because $\varepsilon^{\delta}_{st+v-1} = h^{\delta}_{st+v-1}e^{\delta}_{st+v-1} = \underline{H}'\underline{X}_{st+v-1}e^{\delta}_{st+v-1}$, the state transition Eq (2.1) can be expressed as

$$\underline{X}_{st+v} = A(e_{st+v-1})\,\underline{X}_{st+v-1} + \alpha_0(v)\underline{H} \tag{2.2}$$

where $A(e_{st+v-1}) = B(v) + \underline{\xi}(v)\,\underline{H}'e^{\delta}_{st+v-1}$, Eq (2.2) is the same as the defining equation for multivariate generalized periodic AR process introduced recently by Franses and Paap[9]. However, since Gladychev[10], with periodic time-varying coefficients, it is possible to embed seasons into a multivariate stationary process. More precisely the periodically stationary $\left(\underline{X}_{st+v}\right)$ is equivalent to a stationary process $(\underline{Y}_t)_t$ where $\underline{Y}_t := \left(\underline{X}'_{st+1}, ..., \underline{X}'_{st+s}\right)' \in \mathbb{R}^{sn}$ is a generalized AR process, i.e.,

$$\underline{Y}_t = \Lambda_t \underline{Y}_{t-1} + \underline{e}_t \tag{2.3}$$

where

$$\Lambda_t := \begin{pmatrix} O_{(n)} & \cdots & O_{(n)} & A\left(e_{st}\right) \\ O_{(n)} & \cdots & O_{(n)} & A\left(e_{st+1}\right) A\left(e_{st}\right) \\ \vdots & \ddots & \vdots & \vdots \\ O_{(n)} & \cdots & O_{(n)} & \left\{ \prod_{v=0}^{s-1} A\left(e_{st+s-v-1}\right) \right\} \end{pmatrix}_{sn \times sn},$$

$$\underline{e}_t := \begin{pmatrix} \alpha_0(1)\underline{H} \\ \alpha_0(1)A\left(e_{st+1}\right)\underline{H} + \alpha_0(2)\underline{H} \\ \vdots \\ \sum_{k=1}^{s} \left\{ \prod_{v=0}^{s-k-1} A\left(e_{st+s-v-1}\right) \right\} \alpha_0(k)\underline{H} \end{pmatrix}_{sn \times 1}$$

where, as usual, empty products are set equal to $I_{(n)}$. Since $(e_t)_{t\in\mathbb{Z}}$ is an *i.i.d.* process, stationary and ergodic, $(\Lambda_t, \underline{e}_t)_{t\in\mathbb{Z}}$ is also a stationary and ergodic process and since $E\left\{\log^+ \|\Lambda_1\|\right\} < \infty$ and $E\left\{\log^+ \|\underline{e}_1\|\right\} < \infty$, where $\log^+(x) = \max(\log x, 0)$ for any $x > 0$. The results of this subsection are based on theorems proved by Bougerol and Picard[6].

Theorem 2.1. *Eq (2.3) has a unique strictly stationary and ergodic solution if and only if the top Lyapunov exponent $\gamma_L(\Lambda)$ associated with the sequence of matrices $(\Lambda_t)_t$,*

$$\gamma_L(\Lambda) := \inf_{t>0} \left\{ \frac{1}{t} E\left\{ \log \left\| \prod_{j=0}^{t-1} \Lambda_{t-j} \right\| \right\} \right\} \overset{a.s.}{=} \lim_{t\longrightarrow\infty} \left\{ \frac{1}{t} \log \left\| \prod_{j=0}^{t-1} \Lambda_{t-j} \right\| \right\} \tag{2.4}$$

is strictly negative. The unique stationary solution is ergodic, causal and given by

$$\underline{Y}_t = \sum_{k\geq 0} \left\{ \prod_{j=0}^{k-1} \Lambda_{t-j} \right\} \underline{e}_{t-k} \tag{2.5}$$

where the series (2.5) converges almost surely (a.s.).

Proof. The proof of Theorem 2.1 is similar the proof of Theorem 1.3 by Bougerol and Picard[5]. □

Proposition 2.1. *If*

$$\gamma_L(A) < 0$$

where $\gamma_L(-)$ is the top Lyapunov exponent of the sequence of matrices $\left(\left\{\prod_{v=0}^{s-1} A\left(e_{st+s-v-1}\right)\right\}\right)_t$, then Eq (2.3) has a unique, strictly stationary and ergodic solution given by the series (2.5).

Proof. A simple computation shows that

$$\prod_{j=0}^{t}\Lambda_{t-j}=\Lambda_t\begin{pmatrix} O_{(n)} & \cdots & O_{(n)} & O_{(n)} \\ O_{(n)} & \cdots & O_{(n)} & O_{(n)} \\ \vdots & \ddots & \vdots & \vdots \\ O_{(n)} & \cdots & O_{(n)} & \prod_{j=1}^{t-1}\left\{\prod_{v=0}^{s-1}A\left(e_{s(t-j)+s-v-1}\right)\right\} \end{pmatrix}$$

Therefore, since the top Lyapunov exponent is independent of the norm, by choosing a multiplicative norm it is straighforward to show that $\gamma_L(\Lambda)\leq\gamma_L(A)$. □

Corollary 2.1. *For $P-APGARCH(1,1)$, a sufficient condition which ensures $\gamma_L(A)<0$ is that* $\left\{\prod_{v=0}^{s-1}\left|\gamma_1(v)+\xi_1(v)e_0^{\delta}\right|\right\}<1$.

Proof. If $p=q=1$, we have for all $t\in\mathbb{Z}$, $A(e_{st+v-1})=\gamma_1(v)+\xi_1(v)e_{st+v-1}^{\delta}$ and $\gamma_L(A)=E\left\{\log\left\{\prod_{v=0}^{s-1}\left|\gamma_1(v)+\xi_1(v)e_0^{\delta}\right|\right\}\right\}$. □

The top Lyapunov exponent $\gamma_L(.)$ criterion seems difficult to obtain explicitly, however a potential method to verify whether or not $\gamma_L(.)<0$ is via a Monte-Carlo simulation using Eq (2.3). This fact heavily limits the interest to the criterion in statistical applications. Indeed, the solution need to have some moments to make an estimation theory possible and Condition (2.4) does not guarantee the existence of such moments. Therefore, we have to search for conditions ensuring the existence of moments for the stationary solution, for which the top Lyapunov exponent $\gamma_L(.)$ will be automatically negative.

3 EXISTANCE OF THE HIGHER-ORDER MOMENTS AND COVARIANCE STRUCTURE

In this section, we present a necessary and sufficient conditions for the existence of finite higher-order moments for $P-APGARCH$ process, and to get the covariance structure of the $P-APGARCH$ process.

Theorem 3.1. *Let $(\underline{Y}_t)_t$ be the stationary solution of model (2.3). Assume that $\mu_{\delta m}<\infty$ for any $m>1$.*

1. *If*

$$\rho\left(\prod_{v=0}^{s-1}A_{s-v-1}^{\otimes m}(\mu_\delta)\right)<1 \tag{3.1}$$

then $\underline{Y}_t\in\mathbb{L}_m$.

2. *Conversely, if $\rho\left(\prod_{v=0}^{s-1}A_{s-v-1}^{\otimes m}(\mu_\delta)\right)\geq1$, then there is no strictly stationary solution $(\underline{Y}_t)_t$ to model (2.3) such that $\underline{Y}_t\in\mathbb{L}_m$.*

Proof. **1.** We first define the following $\mathbb{R}^{sn}-$valued stochastic processes

$$\underline{P}_n(t) := \begin{cases} \underline{O}_{(sn)} \text{ if } n < 0 \\ \underline{e}_t + \Lambda_t \underline{P}_{n-1}(t-1) \text{ if } n \geq 0 \end{cases}$$

and for all $n \in \mathbb{Z}$, $\underline{Q}_n(t) = \underline{P}_n(t) - \underline{P}_{n-1}(t)$. It can be easily shown that, for all $n \geq 0$, $\underline{P}_n(t)$ and $\underline{Q}_n(t)$ are measurable functions of $\underline{e}_t, ..., \underline{e}_{t-n}$. Hence the processes $(\underline{P}_n(t))_t$ and $(Q_n(t))_t$ are stationary. From the definition of $\underline{P}_n(t)$ and $\underline{Q}_n(t)$, we can verify that

$$\underline{Q}_n(t) := \begin{cases} \underline{O}_{(sn)} \text{ if } n < 0 \\ \underline{e}_t \text{ if } n = 0 \\ \Lambda_t \underline{Q}_{n-1}(t-1) \text{ if } n > 0 \end{cases}$$

for all $n \in \mathbb{Z}$. Using the properties of Kronecker product, we obtain $\underline{Q}_n^{\otimes m}(t) = \Lambda_t^{\otimes m} \underline{Q}_{n-1}^{\otimes m}(t-1)$ for $m > 1$, $n > 0$ and

$$E\left\{\underline{Q}_n^{\otimes m}(t)\right\} = \left(E\left\{\Lambda_t^{\otimes m}\right\}\right)^n E\left\{\underline{Q}_0^{\otimes m}(t-n)\right\} = \left(E\left\{\Lambda_t^{\otimes m}\right\}\right)^n E\left\{\underline{e}_{t-n}^{\otimes m}\right\}$$

Since $\rho\left(E\left\{\Lambda_t^{\otimes m}\right\}\right) = \rho\left(\prod_{v=0}^{s-1} A_{s-v-1}^{\otimes m}(\mu_\delta)\right) < 1$, we conclude that $\underline{P}_n(t)$ converges in $\mathbb{L}_m$ and almost surely to some limit $\underline{Y}_t \in \mathbb{L}_m$, which is the solution of equation (2.3).

2. From (2.5), we obtain

$$E\left\{\underline{Y}_t^{\otimes m}\right\} \geq \sum_{k \geq 0} E\left\{\underline{Y}_{t,k}^{\otimes m}\right\} = \sum_{k \geq 0} \left(E\left\{\Lambda_t^{\otimes m}\right\}\right)^k E\left\{\underline{e}_{t-k}^{\otimes m}\right\}$$

and the conclusion follows. □

We assume that condition (3.1) holds, this implies that (2.2) has a unique PC solution (in $\mathbb{L}_2$ sense). Taking expectation on both sides of (2.2) and using the notation $\underline{\mu}_m(v) = E\left\{\underline{X}_{st+v}^{\otimes m}\right\}$ and $\underline{\Sigma}_v(h) = E\left\{\underline{X}_{st+v} \otimes \underline{X}_{st+v-h}\right\}$, $m = 1, 2$, gives

$$\underline{\mu}_1(v) = A_v(\mu_\delta)\,\underline{\mu}_1(v-1) + \alpha_0(v)\,\underline{H},\ v \in \{1, ..., s\} \tag{3.2}$$

Recursion (3.2) $s-$times, we get

$$\begin{cases} \underline{\mu}_1(s) = \left(I_{(q)} - \left\{\prod_{v=0}^{s-1} A_{s-v}(\mu_\delta)\right\}\right)^{-1} \sum_{j=0}^{s-1} \left\{\prod_{v=0}^{j-1} A_{s-v}(\mu_\delta)\right\} \alpha_0(v)\,\underline{H} \\ \underline{\mu}_1(v) = \left\{\prod_{j=0}^{v-1} A_{s-j}(\mu_\delta)\right\} \underline{\mu}_1(s) + \sum_{j=0}^{v-1} \left\{\prod_{k=0}^{j-1} A_{s-k}(\mu_\delta)\right\} \alpha_0(v-j)\,\underline{H},\ v \in \{1, ..., s\} \end{cases}$$

The seasonal variance can be obtained as follows

$$\underline{\mu}_2(v) = A_v^{\otimes 2}(\mu_\delta)\,\underline{\mu}_2(v-1) + \underline{\zeta}_v$$

where $\underline{\zeta}_v := \alpha_0^2(v)\,\underline{H}^{\otimes 2} + \alpha_0(v)\left(A_v(\mu_\delta)\otimes\underline{H} + \underline{H}\otimes A_v(\mu_\delta)\right)\underline{\mu}_1(v-1)$, thus

$$\begin{cases} \underline{\mu}_2(s) = \left(I_{(n^2)} - \left\{\prod_{v=0}^{s-1} A_{s-v}^{\otimes 2}(\mu_\delta)\right\}\right)^{-1} \sum_{j=0}^{s-1}\left\{\prod_{v=0}^{j-1} A_{s-v}^{\otimes 2}(\mu_\delta)\right\}\underline{\zeta}_{s-j} \\ \underline{\mu}_2(v) = \left\{\prod_{j=0}^{v-1} A_{s-j}^{\otimes 2}(\mu_\delta)\right\}\underline{\mu}_2(s) + \sum_{j=0}^{v-1}\left\{\prod_{i=0}^{j-1} A_{v-i}^{\otimes 2}(\mu_\delta)\right\}\underline{\zeta}_{v-j} \end{cases}$$

Now, note that for any $h > 0$, we have

$$\begin{aligned} \underline{\Sigma}_v(h) &= E\left\{\underline{X}_{st+v}\otimes\underline{X}_{st+v-h}\right\} \\ &= \left(A_v(\mu_\delta)\otimes I_{(n)}\right)E\left\{\underline{X}_{st+v-1}\otimes\underline{X}_{st+v-h}\right\} + \alpha_0(v)\left(\underline{H}\otimes\underline{\mu}_1(v-h)\right) \\ &= \left(A_v(\mu_\delta)\otimes I_{(n)}\right)\underline{\Sigma}_{v-1}(h-1) + \alpha_0(v)\left(\underline{H}\otimes\underline{\mu}_1(v-h)\right) \\ &= \left\{\prod_{j=0}^{h-1}\left(A_{v-j}(\mu_\delta)\otimes I_{(n)}\right)\right\}\underline{\mu}_2(v-h) \\ &+ \sum_{j=0}^{h-1}\alpha_0(v-k)\left\{\prod_{i=0}^{j-1}\left(A_{v-i}(\mu_\delta)\otimes I_{(n)}\right)\right\}\left(\underline{H}\otimes\underline{\mu}_1(v-j-h)\right) \end{aligned}$$

4 CONCLUSION

This article partially extends $\mathbb{L}_2$ structure of periodic $APGARCH$ model, which allows the volatility of time series to have different dynamics according to the model parameters switching between s regimes. In addition to the conditions ensuring the existence and uniqueness of strictly stationary and second order stationary solution of $P-APGARCH$. We have also given sufficient conditions for the processes to belong $\mathbb{L}_m$, $m \geq 1$, the whole based on a generalized AR representation.

Conflict of Interest

The authors confirm that this article content has no conflict of interest.

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References

[1] Aknouche, A., Bibi, A. (2008) Quasi-maximum likelihood estimation of periodic $GARCH$ and periodic$ARMA-GARCH$ processes. Journal of Time Series Analysis $29\,(1)$, $19-45$.

[2] Bibi, A., Aknouche, A. (2008) On periodic $GARCH$ processes: stationarity, existence of moments and geometric ergodicity. Mathematical Methods of Statistics. $17\,(4)$, $305-316$.

[3] Bollerslev, T. (1986) Generalized autoregressive conditional heteroskedasticity. Journal of Econometrics 31, $307-327$.

[4] Bollerslev, T., Ghysels, E. (1996) Periodic autoregressive conditional heteroskedasticity. Journal of Business & Economic Statistics 14, $139-151$.

[5] Bougerol, P., Picard, N. (1992) Stationarity of $GARCH$ processes and some nonnegative time series. Journal of Econometrics 52, $115-127$.

[6] Bougerol, P., Picard, N. (1992) Strict stationarity of generalized autoregressive processes. Annals of Probability 20,$1714-1730$.

[7] Ding, Z., Granger, W. J., Engle, R. F. (1993) A long memory property of stock market returns and a new model. Journal Empirical Finance 1, $83-106$.

[8] Engle, R. F. (1982) Autoregressive conditional heteroskedasticity with estimates of the variance of U. K. inflation. Econometrica 50, $987-1008$.

[9] Franses, P., Paap, R. (2005) Random coefficient periodic autoregression. Econometric Institute Report EI $2005-2034$.

[10] Gladyshev, E.G. (1961) Periodically correlated random sequences. Soviet Math 2, $385-388$.

[11] Glosten, L. R., Jagannathan, R., Runkle, D. (1993) On the relation between the expected value and the volatility of the nominal excess return on stocks. Journal of Finance 48, $1779-1801$.

[12] Lee, O., Shin, D. W. (2010) Geometric ergodicity and moment conditions for seasonal $GARCH$ model with periodic coefficients. Communications in Statistics - Theory and Methods 39, $38-51$.

[13] Lescheb, I. (2011) Asymptotic inference for periodic $ARCH$ processes. Random Operators and Stochastic Equations 19, $283-294$.